

RELATIVISTIC IDEAL CLOCK

ŁUKASZ BRATEK

INSTITUTE OF NUCLEAR PHYSICS,
POLISH ACADEMY OF SCIENCES,
RADZIKOWSKIEGO 152, PL-31342 KRAKÓW, POLAND
ŁUKASZ.BRATEK@IFJ.EDU.PL

ABSTRACT. Two particularly simple ideal clocks exhibiting intrinsic circular motion with the speed of light and opposite spin alignment are described. The clocks are singled out by singularities of an inverse Legendre transformation for relativistic rotators of which mass and spin are fixed parameters. Such clocks work always the same way, no matter how they move. When subject to high accelerations or falling in strong gravitational fields of black holes, the clocks could be used to test the clock hypothesis.

An ideal clock is a mathematical abstraction of a nearly perfect material clocking mechanism. The clock hypothesis asserts that an ideal clock measures its proper time. This means that the number of consecutive cycles registered by the clock increases steadily with the affine parameter of the worldline of the clock's center of mass (CM). On the dimensional grounds, we may expect that the hypothesis could be violated for extreme accelerations of order $c\omega$ (e.g. $\frac{mc^3}{\hbar} \sim 10^{29} \frac{m}{s^2}$ for the electron).

A recent result [1] suggests that quantum field-theoretical realizations of extended clocks (experiencing the Unruh effect) do not measure their proper times. But the clock hypothesis refers to classical concepts of the relativity theory (e.g. a single worldline), and as such should be first of all tested within the same conceptual framework. A candidate clock should be a relatively simple mathematical device so as to minimize the influence of external disturbances on its structure. If some fundamental limitation were to concern such a clocking standard, the more it would concern more complicated clocks.

A mathematical clock can be devised by an analogy with a quantum particle such as Dirac electron. The intrinsic clock of such particle cannot be impaired – the phase of its wave function oscillates in the rest frame with a fixed frequency determined by only the fundamental constants of nature. But quantum phase is not observable, it would be useless as a clock. Something similar happens with the basic classical analogue of a quantum particle – a structureless material point. The action functional of the material point (to some extent related to a quantum phase) increases linearly with the affine parameter of its worldline. But classical observables are reparameterization invariant, and do not distinguish any particular time variable. In order to play the role of an ideal clock, the material point must be endowed with an additional structure repeatedly changing with the proper time, e.g. connected with some sort of intrinsic rotation. Additionally, for the clock to resemble a quantum particle with its invariable structure as much as possible, it may be required that the clock's mass and magnitude of its spin should have fixed numerical values.

1. A RELATIVISTIC CLOCKING MECHANISM

In the relativity theory, a rotation can be described as a continuous action of an elliptic homography mapping points on a complex plane at one instant to those at another instant and leaving fixed a pair of points: κ_+ and κ_- . It is natural to identify κ_{\pm} with stereographic images $Z(k_{\pm})$ of a pair of null vectors

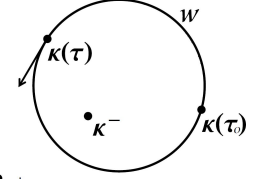
preserved in free motion of any isolated massive system with spin:

$$k_{\pm} \equiv \frac{p}{\sqrt{\langle pp \rangle}} \pm \frac{w}{\sqrt{-\langle ww \rangle}}.$$

Here, p is the momentum vector and w is the Mathisson pseudovector (customarily ascribed to Pauli and Lubański, see [2]). Given a pair k_{\pm} , the homography is set by specifying the motion of a single point $\kappa \equiv Z(k)$ – a stereographic image of a null vector k – about an invariant circle of that homography. The κ is invariant under a local scaling $\delta k = \epsilon k$, so should be the Lagrangian. It is thus necessary that $\langle k\pi \rangle = 0$ identically (with π being the momentum conjugate to k), since otherwise the variation $\delta L = \epsilon(k\partial_k L + \langle k\pi \rangle) + \langle k\pi \rangle \epsilon$ would not be vanishing for arbitrary ϵ . Accordingly, there are two structure constraints:

$$(1) \quad \Psi_3 : \langle k\pi \rangle \simeq 0, \quad \Psi_4 : \langle kk \rangle \simeq 0.$$

Now, it can be deduced what the invariant circle must be. From $w \propto *(p \wedge k \wedge \pi)$ it follows that $\langle kw \rangle = 0$. This means that the image point of k moves about the image circle of w .



As so, k may be thought of as representing the clock's pointer and w as representing the clock's dial (see figure). In free motion in Minkowski spacetime, the vectors k_{\pm} are parallel transported. Then a Lorentz invariant phase can be assigned to κ between instants τ_0 and τ through:

$$(2) \quad \phi(\tau, \tau_0) = i \text{Ln} \left(\frac{\kappa(\tau) - \kappa_+}{\kappa(\tau) - \kappa_-} \cdot \frac{\kappa(\tau_0) - \kappa_-}{\kappa(\tau_0) - \kappa_+} \right).$$

The phase ϕ is a real number. In free motion of the clock, a rotation through $\phi = 2\pi$ represents a single full clocking cycle.¹

2. DYNAMICAL REQUIREMENTS AND THE HAMILTONIAN

The intuition derived from the theory of Eulerian rigid bodies suggests that the the above clock will be insensitive to external influences if both its mass and size is fixed. This requirement can be fulfilled in an invariant way by imposing constraints on the Casimir invariants of the Poincaré group:

$$(3) \quad \Psi_1 : \langle pp \rangle - m^2 \simeq 0, \quad \Psi_2 : \langle ww \rangle + \frac{1}{4} m^4 \ell^2 \simeq 0.$$

Here, constants m and ℓ are fixed parameters with the dimension of mass and that of length. These constraints should be regarded as *primary*, i.e., implied by the form of the Lagrangian.

Motivated by devising an ideal clock, Staruszkiewicz observed [3] that (unlike unitarity) the irreducibility of quantum systems has a classical counterpart realized in postulating Eq.3 as a means to singling out physically appealing Lagrangians. This postulate is in essence equivalent to the earlier, strong conservation idea due to Kuzenko, Lyakhovich and Segal [4], introduced as a basic dynamical principle for devising Lagrangians suitable for geometric models of particles with spin.

As established in Sec.1, the phase space of the simplest clock can be parameterized using components of the position fourvector x and three tangent fourvectors p, k, π bound to satisfy constraints Eqs.1,3, where

$$\langle ww \rangle \equiv -\text{Det} \begin{bmatrix} \langle pp \rangle & \langle pk \rangle & \langle p\pi \rangle \\ \langle kp \rangle & \langle kk \rangle & \langle k\pi \rangle \\ \langle \pi p \rangle & \langle \pi k \rangle & \langle \pi\pi \rangle \end{bmatrix} \simeq \langle pk \rangle^2 \langle \pi\pi \rangle.$$

Between these dynamical variables we assume a Poisson bracket $\{U, V\} \equiv \langle \partial_x U \partial_p V \rangle - \langle \partial_p U \partial_x V \rangle + \langle \partial_k U \partial_\pi V \rangle - \langle \partial_\pi U \partial_k V \rangle$. Eqs.1,3

¹A massless system ($\langle pp \rangle = 0$) would be structurally something different, because for a parabolic homography (preserving a null direction p and an orthogonal to it spatial direction w) the analogous phase is not a Lorentz scalar.

form a system of independent first class constraints with respect to this bracket. In line with Dirac method [5], the most general Hamiltonian is a linear combination of all first class constraints with arbitrary functions u 's as coefficients. It is convenient that the combination be taken as:²

$$(4) \quad H \equiv \frac{u_1}{2m} (\langle pp \rangle - m^2) + \frac{u_2}{2m} \left(\langle pp \rangle + \frac{4}{l^2 m^2} \langle kp \rangle^2 \langle \pi \pi \rangle \right) + u_3 \langle k\pi \rangle + u_4 \langle kk \rangle$$

The equations $\partial_{u_i} H \simeq 0$ form a system of first class constraints equivalent to Eqs.1,3. Next, we introduce velocities $\dot{q} \simeq \partial_p H$ [5]:

$$(5) \quad \begin{aligned} \dot{x} &\simeq (u_1 + u_2) \frac{p}{m} + u_2 \frac{4 \langle kp \rangle \langle \pi \pi \rangle}{l^2 m^3} k \simeq u_1 \frac{p}{m} + u_2 n, \\ \dot{k} &\simeq u_2 \frac{4 \langle kp \rangle^2}{l^2 m^3} \pi + u_3 k, \quad n := \frac{p}{\sqrt{\langle pp \rangle}} - \frac{\sqrt{\langle pp \rangle} k}{\langle kp \rangle}. \end{aligned}$$

By taking projective derivatives, defined recursively by $\mathfrak{d}_t^{n+1} \square := \mathfrak{d}_t(\mathfrak{d}_t^n \square)$, where $\mathfrak{d}_t \square := (\square)_\perp$ and $(\square)_\perp := \square - \frac{\langle p \square \rangle}{\langle pp \rangle} p$, it can be shown that a curvature κ (defined by analogy with Frenet-Serret formulas) is fixed: $\kappa \equiv -(\mathfrak{d}_t x \mathfrak{d}_t x)^{-3} \text{Gram}(\mathfrak{d}_t x, \mathfrak{d}_t^2 x) \simeq 4/l^2$, and that torsion vanishes on account of $\mathfrak{d}_t x$, $\mathfrak{d}_t^2 x$, and $\mathfrak{d}_t^3 x$ being coplanar as p -orthogonal linear combinations of p, k, π . Hence, the trajectory perceived in the CM frame is a circle of a fixed radius $l/2$ (without constraints Eq.3, the radius would vary with the actual state [8]). Correspondingly, the worldline's path is winding up around a fixed space-time cylinder, the main axis of which represents the CM inertial motion. To measure the rate of change of the unit spatial vector n in the CM frame (see Eq.5 for the definition of n), a frequency scalar Ω can be introduced:

$$(6) \quad \Omega^2 := -\frac{\langle \dot{n} \dot{n} \rangle \langle pp \rangle}{\langle p \dot{x} \rangle^2}; \quad \Rightarrow \quad \Omega = \frac{\langle pp \rangle \sqrt{-\langle \dot{k} \dot{k} \rangle}}{|\langle kp \rangle \langle p \dot{x} \rangle|} \text{ if } \dot{p} = 0.$$

For solutions, it reduces to a ratio: $\Omega \simeq (2/l)^{1/2} |u_2/u_1|$, and if $|u_1| > |u_2|$ it is related to a hyperbolic angle Λ between p and x : $\Omega \simeq (2/l) \text{th} \Lambda$. Both Ω and Λ are reparameterization-invariant scalars with obvious physical meaning. On the other hand, Ω and Λ are functions of the arbitrary ratio u_2/u_1 . Thus the motion is indeterminate. To solve this paradox, this ratio needs to be set based on a sound guiding principle, so as not to introduce arbitrary features into the dynamics.

3. SINGULARITIES IN THE INVERSE LEGENDRE TRANSFORMATION

The form of a Lagrangian $L \equiv \langle \dot{x} p \rangle + \langle k \pi \rangle - H$ corresponding to the Hamiltonian Eq.4 is subject to invertibility of the map Eq.5 restricted to a submanifold determined by the constraints Eqs.3,1. For the purpose of the invertibility analysis, it must suffice to focus upon Lorentz scalars only. On the submanifold of interest, we may consider a map between two sets of variables: $u_1, u_2, u_3, \langle kp \rangle, \langle p \pi \rangle$ and $\langle \dot{k} \dot{k} \rangle, \langle \dot{k} \dot{x} \rangle, \langle \dot{x} \dot{x} \rangle, \langle \dot{k} \dot{x} \rangle, \langle \dot{k} \dot{k} \rangle$:

$$(7) \quad \begin{aligned} \langle \dot{x} \dot{x} \rangle &\simeq u_1^2 - u_2^2, & \langle \dot{k} \dot{x} \rangle &\simeq \frac{\langle kp \rangle}{m} (u_1 + u_2), \\ \langle \dot{k} \dot{x} \rangle &\simeq \frac{\langle kp \rangle}{m} \left(\frac{4 \langle kp \rangle \langle p \pi \rangle}{m^3 l^2} u_2 + u_3 \right) (u_1 + u_2), \\ \langle \dot{k} \dot{k} \rangle &\simeq -\frac{4 \langle kp \rangle^2}{l^2 m^2} u_2^2, & \langle \dot{k} \dot{k} \rangle &\simeq 0. \end{aligned}$$

The number of new constraints for velocities depends on the rank of the Jacobian matrix of this map. Non-zero minors of maximal rank 4 for this Jacobian are: $j_1 = \frac{16 \langle kp \rangle^3}{l^2 m^4} u_2 (u_1 + u_2) (u_2^2 - u_1^2)$

² The original KLS Hamiltonian [4] involved a complex variable ζ , ($\zeta \equiv z(k)$), inherited from a primary Lagrangian. Starting with a related Lagrangian expressed in terms of k , a Hamiltonian analogous in form to Eq.4 was arrived at in [6] (upon earlier reducing an extended phase space). Our approach goes in the opposite direction. We start with a Hamiltonian deduced from first principles. In [7] we generalized this method onto systems described by a collection of fourvectors.

and $j_2 = \frac{4 \langle kp \rangle}{l^2 m^3} u_2 j_1$. Since $\langle kp \rangle \neq 0$ (for a timelike p and a null k), this implies that the Jacobian rank, Rk , is dependent on $u_{1,2}$. Full analysis distinguishes the following 4 regimes:

	Rk		velocity constraints
i)	4	$u_1^2 \neq u_2^2 \neq 0$	$\langle \dot{k} \dot{k} \rangle \simeq 0$
ii)	3	$u_1 = u_2 \neq 0$	$\langle \dot{k} \dot{k} \rangle \simeq 0, \langle \dot{x} \dot{x} \rangle \simeq 0, l^2 \langle \dot{k} \dot{k} \rangle + \langle \dot{k} \dot{x} \rangle^2 \simeq 0$
iii)	2	$u_1 = -u_2 \neq 0$	$\langle \dot{k} \dot{k} \rangle \simeq 0, \dot{x} \propto k \Rightarrow \langle \dot{x} \dot{x} \rangle \simeq 0$
ii')	3	$u_2 = 0, u_1 \neq 0$	$\langle \dot{k} \dot{k} \rangle \simeq 0, \langle \dot{k} \dot{x} \rangle \simeq 0$

The ii' case will not be of concern here, and $u_2 \neq 0$ is assumed from now on. To find explicit expressions for momenta, two cases are to be considered: $u_1 + u_2 \neq 0$ / i, ii/ or $u_1 + u_2 = 0$ / iii/.

• For $u_2(u_1 + u_2) \neq 0$ the ansatz $p = \alpha_1 \dot{x} + \alpha_2 k$ and $\pi = \beta_1 \dot{k} + \beta_2 k$ allows to express momenta in terms of velocities and u 's. On substituting to Eq.5 and solving for $\alpha_{1,2}, \beta_{1,2}$, one gets:

$$(8) \quad \begin{aligned} p &= \frac{m}{u_1 + u_2} \dot{x} - \frac{l^2 m (u_1 + u_2)^2 (\langle \dot{k} \dot{k} \rangle - 2 \langle \dot{k} \dot{x} \rangle u_3)}{4 \langle \dot{k} \dot{x} \rangle^2 u_2} \frac{k}{\langle \dot{k} \dot{x} \rangle}, \\ \pi &= \frac{l^2 m (u_1 + u_2)^2}{4 \langle \dot{k} \dot{x} \rangle^2 u_2} (\dot{k} - k u_3). \end{aligned}$$

The Ψ_3 constraint leads to $\langle \dot{k} \dot{k} \rangle \simeq 0$ (consistently with Ψ_4), while the $\Psi_{1,2}$ constraints give conditions for $u_{1,2}$:

$$\frac{1}{(u_1 + u_2)^2} \langle \dot{x} \dot{x} \rangle + \frac{u_1 + u_2}{2 u_2} \xi = 1 \quad \wedge \quad \frac{(u_1 + u_2)^2}{4 u_2^2} \xi = 1. \quad \xi := -l^2 \frac{\langle \dot{k} \dot{k} \rangle}{\langle \dot{k} \dot{x} \rangle^2}$$

The resulting $u_{1,2}$ can be expressed as independent functions of velocities, provided that the Jacobian determinant $\frac{\partial(\Psi_1, \Psi_2)}{\partial(u_1, u_2)}$, equal to $\frac{-m^6 l^2 \xi}{4 u_2^3 (u_1 + u_2)} \langle \dot{x} \dot{x} \rangle$, is nonzero, which leads to a *Lagrangian of the first kind*. Otherwise, if $\langle \dot{x} \dot{x} \rangle = 0$, then one gets $u_1 = u_2$ and a frequency constraint $\langle \dot{k} \dot{x} \rangle^2 + l^2 \langle \dot{k} \dot{k} \rangle \simeq 0$. This leads to a *Lagrangian of the second kind*.

• For $u_2 \neq 0$ and $u_1 + u_2 = 0$, one is led to a *Lagrangian of the third kind* with $\dot{x} \propto k$ (when $\langle \dot{x} \dot{x} \rangle \simeq 0, \langle \dot{k} \dot{x} \rangle \simeq 0$ and $\langle \dot{k} \dot{k} \rangle \simeq 0$).

3.1. Null worldlines principle. Above, the rank of the inverse Legendre transformation, qualitatively discriminated between two separate regimes: $\langle \dot{x} \dot{x} \rangle \neq 0$ (maximal rank) and $\langle \dot{x} \dot{x} \rangle = 0$ (lower ranks). Now, two other premises can be brought to the attention, as to why specifically the condition $\langle \dot{x} \dot{x} \rangle = 0$ is so particular.

In the maximal rank case, assuming any constraints such that $\langle \dot{x} \dot{x} \rangle \neq 0$ would be a matter of arbitrary decision. For $\langle \dot{x} \dot{x} \rangle > 0$, choosing a given function for Ω is equivalent to setting the hyperbolic angle Λ . But there is no privileged hyperbolic angle in the (homogeneous) Lobachevsky space of fourvelocities (a similar argument on de Sitter hyperboloid would apply to the 'tachionic' sector $u_2^2 > u_1^2$). On the contrary, null worldlines are distinguished by the lightcone structure of the spacetime. With $\langle \dot{x} \dot{x} \rangle = 0$, the velocity can be fixed in a manifestly relativistically invariant manner independently of parameterization. We stress this important circumstance, since outside the light cone, a more general condition $\langle \dot{x} \dot{x} \rangle = \sigma$ with a given nonzero function σ , neither would set a velocity nor be reparameterization invariant. This qualitative difference should find its reflection also in the structure of the respective Lagrangians.³ Yet, there is an insightful remark due to Dirac, showing the distinguished role of the condition $\langle \dot{x} \dot{x} \rangle = 0$. It is a [counterintuitive] consequence of

³ This difference is already seen for a material point described by a general Lagrangian $L = \frac{1}{2} (w^{-1} \langle \dot{x} \dot{x} \rangle + m^2 w)$. The equation $\partial_w L = 0$ implies two qualitatively distinct regimes: 1) in which w is a function of \dot{x} , then $w = m^{-1} \sqrt{\langle \dot{x} \dot{x} \rangle}$, and 2) in which w is independent of \dot{x} , requiring $m = 0$ and a constraint $\langle \dot{x} \dot{x} \rangle = 0$. The resulting Lagrangians are: 1) that of a massive particle $L = m \sqrt{\langle \dot{x} \dot{x} \rangle}$ with $p = m \frac{\dot{x}}{\sqrt{\langle \dot{x} \dot{x} \rangle}}$ and 2) that of a massless particle $L = \frac{1}{2} w^{-1} \langle \dot{x} \dot{x} \rangle$ with $p = w^{-1} \dot{x}$ and an arbitrary w transforming as $\delta w = w \delta \epsilon$ under a reparameterization $\delta \dot{x} = \dot{x} \delta \epsilon$. The analytic form of $L = m \sqrt{\langle \dot{x} \dot{x} \rangle}$ would not be suitable in a region containing the surface $\langle \dot{x} \dot{x} \rangle = 0$, where the corresponding p would be divergent.

Dirac equation, that a measurement of the electron's instantaneous motion is certain to give the speed of light, which Dirac mentions in his *Principles* [9] and asserts this result to be generally true in a relativistic theory.

The Dirac observation in conjunction with previous findings tempts one to conjecture that worldlines of classical analogs of quantum elementary particles should be null.

4. LAGRANGIANS OF THE FIRST KIND

In the sub-luminal sector ($u_1^2 > u_2^2$) let $u_1 = \rho \cosh \psi$, $u_2 = \rho \sinh \psi$, $\rho > 0$, $|\psi| < \infty$. Then from Eq.7: $\rho = \sqrt{\langle \dot{x}\dot{x} \rangle}$, $\tanh \psi = \frac{\sqrt{\langle \dot{x}\dot{x} \rangle}}{2 \pm \sqrt{\langle \dot{x}\dot{x} \rangle}}$. With the resulting $u_{1,2}$ substituted in Eq.8, two Lagrangians follow: $\tilde{L} = m\sqrt{\langle \dot{x}\dot{x} \rangle} \sqrt{1 \pm \sqrt{\langle \dot{x}\dot{x} \rangle}} + \lambda_1 \langle kk \rangle + \lambda_2 \langle \dot{k}\dot{k} \rangle$ ($\lambda_{1,2}$ involve arbitrary $u_{3,4}$).

In the super-luminal sector ($u_1^2 < u_2^2$) – which may be considered on account of x not being assigned to a CM motion – a similar analysis (with $u_1 = -\hat{\rho} \sinh \psi$ and $u_2 = -\hat{\rho} \cosh \psi$, $\hat{\rho} \neq 0$) leads to a single Lagrangian $\tilde{L} = m\sqrt{-\langle \dot{x}\dot{x} \rangle} \sqrt{\langle \dot{x}\dot{x} \rangle - 1} + \lambda_1 \langle kk \rangle + \lambda_2 \langle \dot{k}\dot{k} \rangle$.

In both cases, the last term in \tilde{L} (whose only effect is an additive gauge-like term in the canonical momentum $\partial_{\dot{k}} \tilde{L} \rightarrow \partial_{\dot{k}} \tilde{L} + \alpha k$) can be integrated off by parts. On denoting the remaining term $(\lambda_1 - (1/2)\lambda_2)\langle kk \rangle$ by $\lambda \langle kk \rangle$, one finally ends up with two Lagrangians (equivalent to those arrived at in [4, 3]):

$$(9) \quad L_{\pm} = m \sqrt{\langle \dot{x}\dot{x} \rangle} \left(1 \pm \sqrt{-\ell^2 \frac{\langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle^2}} \right) + \lambda \langle kk \rangle,$$

with their respective Lagrange multipliers λ . The sub-luminal Lagrangian L_+ is that of the Fundamental Relativistic Rotator [3]. With the Lagrangian L_- we could consider both sub- or super-luminal motions.

In the clock context, it is appropriate to recall an earlier result [10] published in [11] that the Lagrangians Eq.9 can be alternatively arrived at by adopting a physically dubious condition that the Hessian matrix $\partial_{\dot{q}}^2 L$ for a general Lagrangian $L = f(\xi)\sqrt{\langle \dot{x}\dot{x} \rangle}$ expressed in terms of only the 5 degrees of freedom characteristic of a rotator – Cartesian $\mathbf{x}(t)$ and spherical $\vartheta(t), \varphi(t)$ (considered as functions of $x^0 \equiv t$) – must be zero. As shown therein, this leads to a differential equation for f : $f_{,\xi} f + 2\xi(f_{,\xi})^2 + f_{,\xi\xi} f = 0$. As a direct consequence of this, the clocking frequency becomes indeterminate. This conforms with what has been concluded in Sec.2. For reasons described in Sec.3.1, with the Lagrangian Eq.9, there would be no privileged velocity constraint suitable to set this frequency so as to make the motion determinate, while conditions involving $\langle \dot{x}\dot{x} \rangle = 0$ (such as *ii* or *iii*) would not be compatible with the analytic structure of these Lagrangians (the canonical momenta $\partial_{\dot{q}} L$ would involve indeterminate forms 0/0). For these reasons we must come to the conclusion that Eq.9 does not describe a clock at all.

It seems that neither considering more complicated systems [12, 7] nor introducing interactions [11] would help to remove this indeterminacy of motion. For example, in the electromagnetic field, the consistency requirements $\{\Psi_{1,2}, H\} \simeq 0$ (with $p - eA$ substituted for p by the minimal coupling principle) lead to a secondary constraint $F_{\mu\nu} p^{\mu} k^{\nu} \simeq 0$, which for rotators reduces to a condition $F_{\mu\nu} \dot{x}^{\mu} k^{\nu} = 0$ strictly connected with the Hessian singularity alluded to above. Although this condition might lead to a unique motion in some situations (e.g. with appropriate initial data in a uniform magnetic field [13]) this may not be so in general (see, a toy model [14]).

5. IDEAL CLOCKS

5.1. Second kind Lagrangian. The new velocity constraints arranged to forms of the first degree in the velocities read:

$$(10) \quad \frac{\langle \dot{x}\dot{x} \rangle}{\langle \dot{k}\dot{k} \rangle} \simeq 0, \quad \ell^2 \frac{\langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle} + \langle k\dot{x} \rangle \simeq 0.$$

By eliminating these constraints from Eq.7, one finds $u_1 = \chi$, $u_2 = \chi$, $u_3 = v$, $\langle kp \rangle = \frac{m\langle \dot{x}\dot{x} \rangle}{2\chi}$ and $\langle p\dot{x} \rangle = \frac{\ell^2 m^2}{2\langle \dot{k}\dot{k} \rangle} \left(\frac{\langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle} - v \right)$, where χ and v are arbitrary functions. After discarding a total derivative involving $\langle k\dot{k} \rangle$ and the higher order terms in the velocity constraints (irrelevant for the Dirac variational procedure [5]), the resulting Lagrangian can be arranged in a form with a new independent variable $\kappa(\chi) \equiv \frac{\langle kp \rangle}{m}$ and a Lagrange multiplier λ :

$$(11) \quad L = \frac{m\kappa}{2} \frac{\langle \dot{x}\dot{x} \rangle}{\langle \dot{k}\dot{k} \rangle} + \frac{m}{4\kappa} \left(\ell^2 \frac{\langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle} + \langle k\dot{x} \rangle \right) + \lambda \langle kk \rangle.$$

As expected, this Lagrangian is linear in the velocity constraints, with functions of momenta as coefficients. In view of the equation $\partial_{\kappa} L = 0$, the conditions Eq.10 can be regarded as consequences of one another, and hence, only $\langle \dot{x}\dot{x} \rangle = 0$ may be imposed as a subsidiary condition. Then κ becomes arbitrary. Conversely, if $\partial_{\kappa} L = 0$ is to be satisfied for arbitrary κ , then both conditions in Eq.10 follow. The Casimir invariants $\langle pp \rangle = \frac{m^2}{2}(1 + \xi)$ and $\langle uw \rangle = -\frac{\ell^2 m^4}{4} \xi$ are bound to satisfy only a single constraint $\langle pp \rangle \simeq \frac{m^2}{2} - \frac{2}{\ell^2 m^2} \langle uw \rangle$ and off the surface $\frac{\ell^2 \langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle} + \langle k\dot{x} \rangle = 0$ they would be functions of the velocities. But for Eq.11 the principal conditions are satisfied on the basis of Hamilton's principle, either supplemented with the null worldlines principle or with the condition that κ be independent of the velocities.⁴ The latter requirement is crucial, since otherwise, by solving $\partial_{\kappa} L = 0$ for κ , one would end up with a qualitatively different Lagrangian $m\sqrt{\frac{\langle \dot{x}\dot{x} \rangle}{2}} \left(1 + \ell^2 \frac{\langle \dot{k}\dot{k} \rangle}{\langle \dot{x}\dot{x} \rangle^2} \right) + \lambda \langle kk \rangle$ whose analytic form is not admissible on the surface Eq.10 (the momenta $\partial_{\dot{q}} L$ would involve indeterminate forms $\frac{0}{0}$).

5.1.1. Connection with a family of Relativistic Rotators. To extend the construction in [3] so as to include also the case of Sec.5.1, let a class of projection invariant Lagrangians of the first degree in the velocities be considered, whose form would be admissible also on the surface $\langle \dot{x}\dot{x} \rangle = 0$ and compatible with the condition $\langle k\dot{x} \rangle \neq 0$:

$$(12) \quad L_{\mathcal{F}} = \frac{m\kappa}{2} \frac{\langle \dot{x}\dot{x} \rangle}{\langle \dot{k}\dot{k} \rangle} + \frac{m}{2\kappa} \langle k\dot{x} \rangle \mathcal{F}(\xi) + \lambda \langle kk \rangle.$$

The κ must have appeared in this precise way for the dimensional grounds and it must transform as $\kappa \rightarrow \alpha \kappa$ when $k \rightarrow \alpha k$ on account of the assumed projection invariance. Here, \mathcal{F} is any function. If $\partial_{\kappa} L_{\mathcal{F}} = 0$, the principal constraints reduce to $\mathcal{F}(\xi) - 2\xi \mathcal{F}'(\xi) = 1$ and $4\xi \mathcal{F}'(\xi)^2 = 1$ for any κ . If κ is not a function of velocities, then $\partial_{\kappa} L_{\mathcal{F}} = 0$ implies $\mathcal{F} = 0$ (and $\langle \dot{x}\dot{x} \rangle = 0$), then the principal conditions give $\mathcal{F}' = -\frac{1}{2}$ and $\xi = 1$. Hence, to a linear order, $\mathcal{F}(\xi) = (1 - \xi)/2 + o(1 - \xi)$ in the vicinity of $\xi = 1$. And this is another way of arriving at Eq.11. In contrast, for κ not independent of the velocities, one would conclude from $\partial_{\kappa} L_{\mathcal{F}} = 0$ that $\kappa = \langle k\dot{x} \rangle \sqrt{\mathcal{F}(\xi)/\langle \dot{x}\dot{x} \rangle}$ and end up with a class of Lagrangians $m\sqrt{\langle \dot{x}\dot{x} \rangle} \mathcal{F}(\xi)$ describing relativistic rotators considered in [3] (which includes Lagrangians L_{\pm} of Sec.4 as a special case with $\mathcal{F}(\xi) = 1 \pm \sqrt{\xi}$).

5.2. Third kind Lagrangian. Putting $u_1 = -u_2$, consider for a while a restricted Legendre transformation with p left unaltered. Taking $\pi = \mp \frac{\ell m^2}{2} \frac{k - k u_3}{\langle kp \rangle \sqrt{-\langle \dot{k}\dot{k} \rangle}}$ and $u_2 = \mp \frac{\ell m}{2\langle kp \rangle} \sqrt{-\langle \dot{k}\dot{k} \rangle}$ implied by Eqs.5,7 into account (where $\text{sgn}(\frac{\langle p\dot{x} \rangle}{\langle kp \rangle}) = \pm 1$) and integrating off the term linear in $\langle kk \rangle$, one arrives at a Lagrangian:

$$(13) \quad L = \langle \dot{x}p \rangle \pm \frac{\ell m^2}{2} \frac{\sqrt{-\langle \dot{k}\dot{k} \rangle}}{\langle kp \rangle} + \lambda \langle kk \rangle.$$

⁴Because $\langle k\partial_{\dot{x}} L \rangle \equiv m\kappa$, the freedom in choosing κ at an instant (with k being set) involves the freedom in choosing a combination of momentum variables in p . This dependence of a Lagrangian on momentum variables is characteristic of systems with velocity constraints [5].

By making arbitrary variations w.r.t. p (δL must be independent of δp [5]), the result $\dot{x} = \pm \frac{lm^2}{2} \frac{\sqrt{-\langle \dot{k}\dot{k} \rangle}}{\langle kp \rangle^2} k$ following from Eqs.5,7 can be re-obtained. It implies $\langle \dot{x}e \rangle = \pm \frac{lm^2}{2} \frac{\sqrt{-\langle \dot{k}\dot{k} \rangle}}{\langle kp \rangle^2} \langle ek \rangle$ for any vector e , and this fact can be used to eliminate p from L . Hence, the alternative Lagrangian takes on a form involving arbitrary e such that $\langle ke \rangle \neq 0$:

$$(14) \quad L = m \sqrt[4]{-\frac{4\ell^2 \langle \dot{x}e \rangle^2 \langle \dot{k}\dot{k} \rangle}{\langle ek \rangle^2}} + \lambda \langle kk \rangle.$$

For $\langle ek \rangle$ to be nonzero, it would suffice that e be obtained from any timelike vector by a two-parameter transformation group $e \rightarrow \alpha(e + \beta k)$, with α, β being arbitrary functions. This freedom in choosing e must be physically irrelevant, and this will be so if $\partial_e L = 0$. This implies $\dot{x} = \frac{\langle \dot{x}e \rangle}{\langle ke \rangle} k$. Furthermore, $p := \partial_{\dot{x}} L$ is collinear with e and is independent of the scale of e . As so, p may be substituted in place of e in the expression for $\partial_{\dot{x}} L$, hence

$$(15) \quad \frac{2}{\ell} = \frac{m^2 \sqrt{-\langle \dot{k}\dot{k} \rangle}}{|\langle \dot{x}p \rangle \langle kp \rangle|} \Rightarrow \Omega = \frac{2}{\ell} \frac{\langle pp \rangle}{m^2} \quad (\text{from Eq.6}).$$

The constraint $\langle pp \rangle - m^2 \simeq 0$ does not follow from the Lagrangians Eqs.13,14, nevertheless it is essential for consistency with the map Eq.7. It must be regarded as a *secondary* first class constraint (whose purpose is to set Ω to $2/\ell$ and the orbital radius to $\ell/2$, consistently with the equations of motion).

5.3. Comparison of the clocks. It is convenient to write down the Hamiltonian equations in the CM gauge: $\langle p\pi \rangle = 0$, $\langle kp \rangle = m$, $\langle p\dot{x} \rangle = m$ and to consider a unit space-like direction n defined in Eq.5, which is collinear with the projection of k onto a subspace orthogonal to p . Together with the consistency requirements $0 = \langle \langle kp \rangle, H \rangle$ and $0 = \langle \langle p\pi \rangle, H \rangle$, this implies for $u_1 = \pm u_2$ that $u_1 = 1$, $u_2 = \pm 1$, $u_3 = 0$ and $u_4 = \pm \frac{m}{2}$. This way the Hamiltonian equations reduce to

$$\dot{x} = \frac{p}{m} \mp n, \quad \dot{p} = 0, \quad \dot{n} = \pm \frac{4}{m\ell^2} \pi, \quad \dot{\pi} = \mp m n,$$

with $\langle m \rangle = -1$, $\langle np \rangle = 0$ (then $k = \frac{p}{m} + n$). The equations for \dot{n} and $\dot{\pi}$ imply a uniform motion with frequency $\Omega = \frac{2}{\ell}$ about a great circle on the unit sphere: $\ddot{n} + \frac{4}{\ell^2} n = 0$. The null directions of the clocks' velocity vectors \dot{x} are conjugate to one another by the reflection $\frac{m\dot{x}}{\langle p\dot{x} \rangle} \rightarrow \frac{2p}{m} - \frac{m\dot{x}}{\langle p\dot{x} \rangle}$. Interestingly, the two clocks have opposite spin alignment:

$$p \wedge k \wedge \pi = \pm \frac{m\ell^2}{4} p \wedge n \wedge \dot{n} = \pm m p \wedge x \wedge \dot{x},$$

where $x = \frac{p}{m}t + \frac{\ell}{2}r(\varphi)$, $\langle rr \rangle = -1$, $\langle pr \rangle = 0$, $\varphi = \frac{2}{\ell}t$ (then $n = \pm r'(\varphi)$). In a sense, the two clocks can be regarded as a limiting case of the Lagrangian Eq.9 when $\langle \dot{x}\dot{x} \rangle \rightarrow 0$ with: a) $\frac{\ell^2 \langle \dot{k}\dot{k} \rangle}{\langle k\dot{x} \rangle^2} \rightarrow -1$ for the clock Eq.11 or b) $\langle k\dot{x} \rangle \rightarrow 0$ for the clock Eq.14.

6. SUMMARY AND FUTURE APPLICATIONS

In this paper were described two mathematical clocks which are relativistic rotators exhibiting intrinsic circular motion with the speed of light and opposite spin alignment. The Lagrangians of the clocks were distinguished by a singularity of an inverse Legendre map for rotators of which Casimir scalars are fixed parameters. Such clocks are perfect, they work always the same way, no matter how they move.

In future works, the two ideal clocks can be used to test the clock hypothesis. In free motion, the phase associated with the intrinsic circular motion of these clocks increases steadily with the affine parameter of the center of mass (CM). But it is not a priori obvious (even in the limit $\ell \rightarrow 0$) if this property will survive for accelerated motions of the CM, e.g. for a constrained

motion along a strongly curved worldline. For such motions, the *chronometric curve* – that is, properly parameterized helical null worldline of an ideal clock – would undergo additional distortions and this could affect the steady clocking rate.

Testing the clock hypothesis requires introducing interactions. However, usual coupling with external fields may lead to inconsistencies, see [15]. In this context, it would be instructive to see the implications of the secondary constraint $F_{\mu\nu} p^\mu k^\nu \simeq 0$ appearing when ideal clocks are minimally coupled with the electromagnetic field. Furthermore, it would be interesting to study the motion in strong gravitational fields of black holes. In curved spacetimes, there might arise problems even with defining the rotation phase: when global teleparallelism is lost, the local reference frames, used to measure the infinitesimal phase increments at various instants, cannot be unambiguously connected; in addition, some disturbances in the phase could appear due to rotation of local inertial frames.

REFERENCES

- [1] K. Lorek, J. Louko, and A. Dragan, "Ideal clocks – a convenient fiction," *Classical and Quantum Gravity*, vol. 32, no. 17, p. 175003, 2015.
- [2] T. Sauer and A. Trautman, "Myron Mathisson: What little we know of his life," *Acta Phys. Pol. B Proc. Suppl.*, vol. 1, pp. 7–26, Feb. 2008. International Conference Myron Mathisson: his life, work and influence on current research, Stefan Banach International Mathematical Center, Warsaw, Poland, 18–20 October 2007.
- [3] A. Staruszkiewicz, "Fundamental relativistic rotator," *Acta Phys. Pol. B Proc. Suppl.*, vol. 1, pp. 109–112, 2008. Presented at the conference Myron Mathisson: his life, work and influence on current research, Stefan Banach International Mathematical Center, Warsaw, Poland, 18–20 October 2007.
- [4] S. M. Kuzenko, S. L. Lyakhovich, and A. Y. Segal, "a Geometric Model of the Arbitrary Spin Massive Particle," *International Journal of Modern Physics A*, vol. 10, pp. 1529–1552, 1995.
- [5] P. A. Dirac, "Generalized Hamiltonian dynamics," *Can.J.Math.*, vol. 2, pp. 129–148, 1950.
- [6] S. Das and S. Ghosh, "Relativistic spinning particle in a noncommutative extended spacetime," *Phys. Rev. D*, vol. 80, p. 085009, Oct 2009.
- [7] Ł. Bratek, "Spinor particle. An indeterminacy in the motion of relativistic dynamical systems with separately fixed mass and spin," *Journal of Physics Conference Series*, vol. 343, p. 012017, Feb. 2012.
- [8] Ł. Bratek, "Can rapidity become a gauge variable? dirac hamiltonian method and relativistic rotators," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 5, p. 055204, 2012.
- [9] P. A. M. Dirac, *The Principles of Quantum Mechanics*, vol. 27 of *The international series of monographs on physics*. fourth revised reprinted ed., 2009.
- [10] Ł. Bratek, "On the nonuniqueness of free motion of the fundamental relativistic rotator," *ArXiv e-prints 0902.4189*, Feb. 2009.
- [11] Ł. Bratek, "Fundamental relativistic rotator: Hessian singularity and the issue of the minimal interaction with electromagnetic field," *Journal of Physics A Mathematical General*, vol. 44, p. 195204, May 2011.
- [12] Ł. Bratek, "Breathing Relativistic Rotators and Fundamental Dynamical Systems," *J. Phys.*, vol. A43, p. 015208, 2009.
- [13] V. Kassandrov, N. Markova, G. Schäfer, and A. Wipf, "On a model of a classical relativistic particle of constant and universal mass and spin," *Journal of Physics A Mathematical General*, vol. 42, Aug. 2009.
- [14] Ł. Bratek, "False constraints. A toy model for studying dynamical systems with degenerate Hessian form," *Journal of Physics A Mathematical General*, vol. 43, p. 5206, Nov. 2010.
- [15] P. A. M. Dirac, "A positive-energy relativistic wave equation," vol. 332, pp. 435–445, May 1971.